J. Fluid Mech. (1981), *vol.* 108, *pp.* 43–53
Printed in Great Britain

The importance of discharge siting upon contaminant dispersion in narrow rivers and estuaries

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(Received 10 March 1980 and in revised form 17 September 1980)

Far downstream of a sudden contaminant release in a narrow channel the concentration depends on the cloud size. This is largely determined by the longitudinal shear dispersion and the time of travel of the cloud. Near the source the efficiency of the shear dispersion and the velocity of the cloud are strongly dependent upon the source location across the flow. The shear dispersion is greatest when there is both strong shear and strong turbulent mixing (i.e. away from either the centre-line or the banks), while the velocity is least and the time-lag maximized for a source on the banks. The quantitative influence far downstream can be characterized in terms of a deficit variance and a centroid displacement. In this paper exact results are derived for these quantities. It is shown that, except when the banks are extremely steep, the time-lag has the strongest effect and the concentration far downstream of a point discharge is minimized when the discharge is sited at the bank.

1. Introduction

Contaminant releases into rivers or estuaries are an almost inevitable feature of many agricultural, domestic and industrial practices. By default, these discharges usually take place at the river bank. The question to which the present work was directed was to what extent the environmental impact of a discrete contaminant release could be reduced by moving the point of discharge away from the bank. The unexpected answer is that, unless the banks are extremely steep, the bank is the best site for a point discharge in narrow rivers and estuaries.

First it is necessary to define a quantitative criterion for environmental impact. Clearly, the greatest concentrations will be met close to the discharge where the contaminant has not yet become well-mixed across the flow (see figure 1). However, for narrow rivers and estuaries this region is quite localized to the immediate vicinity of the discharge. Thus, we are led to consider the concentration far downstream of the discharge.

In the far field the contaminant is well mixed across the flow and the dispersion is primarily in the longitudinal direction. Moreover, the concentration distribution along the flow is described by a constant coefficient diffusion equation (Taylor 1953; Fischer 1967). In particular, the variance grows at a constant rate and the centroid moves at the bulk velocity. Hence, the effects of the detailed discharge conditions can be characterized in terms of a deficit variance and a centroid displacement (i.e. in the early stages before the contaminant cloud has become well-mixed across the flow the

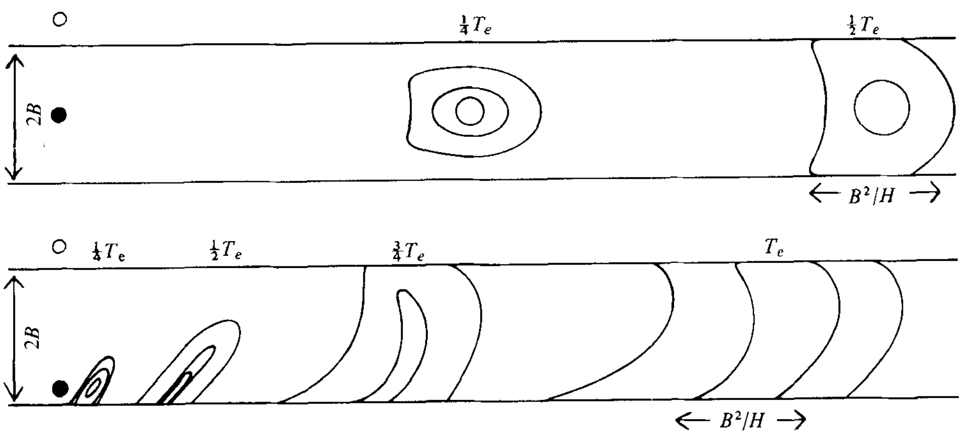

FIGURE 1. Sketch of contaminant clouds at equal time intervals for two different discharge positions across the flow. The e-folding time T_e for mixing across the flow scales as B^2/Hu_* and typically corresponds to a hundred channel breadths downstream.

shear dispersion is not yet fully efficient, and the velocity of the cloud is different from the bulk velocity). At a fixed distance downstream of the discharge position, the time of arrival of the contaminant cloud depends upon the centroid displacement. Thus, as a quantitative measure of the importance of the discharge conditions we employ the 'adjusted deficit variance', with the time-lag incorporated (see appendix). The smaller the adjusted deficit variance, the smaller the contaminant concentration at all large distances downstream of the discharge.

It is this two-component structure of the adjusted deficit variance that makes it difficult to make intuitive predictions of the effects of change discharge conditions. For example, shear dispersion depends upon the combination of cross-stream mixing and velocity shears. At the banks the mixing is weak, while at the centre-line the shear is weak. Thus, we can infer that the shear dispersion will be most efficient, and therefore the deficit variance least, when the discharge is away from either the centre-line or the sides. However, the time-lag (i.e. centroid displacement) is greatest when the discharge is in the slow-moving water at the bank (see figure 1). To make a prediction for the adjusted deficit variance we need to know the relative importance of the two contributions. The subsequent mathematical analysis reveals that the time-lag tends to dominate. In particular, the bank is the best site for a point discharge unless the channel is extremely steep-sided.

2. Moment equations

Vertical mixing of a contaminant takes place within a few tens of water depths downstream of the discharge. This is to be contrasted with the hundred or so channel breadths in which lateral mixing takes place (Smith 1979). Thus, for simplicity we shall average out the vertical structure and take the contaminant diffusion equation to have the two-dimensional form

$$\left.\begin{aligned} h\partial_t c + (u-\bar{u})\,h\partial_x c - hK_1\,\partial_x^2 c - \partial_y(hK_2\,\partial_y c) &= 0, \\ \text{with}\qquad hK_2\,\partial_y c = 0 \quad \text{at} \quad y &= y_-, y_+, \\ \text{and}\qquad c = q(y)\,\delta(x) \quad \text{at} \quad t &= 0. \end{aligned}\right\} \tag{2.1}$$

Here we have used axes moving with the bulk velocity $\bar{u}$, $h(y)$ is the water depth, $u(y)$ longitudinal velocity, $K_1(y)$, $K_2(y)$ are horizontal dispersion coefficients, y_+, y_- the sides of the channel, and $q(y)$ is the source distribution. The effect of the vertical velocity shear is to make the local longitudinal dispersion coefficient K_1 be about a factor of 40 larger than the transverse term K_2 (Elder 1959). For estuaries the neglect of tidal variations is justifiable only if the width is less than about 50 m (Smith 1979, equation 2.16).

The centroid displacement and deficit variance are direct properties of the first and second moments of the concentration distribution. Thus, following Aris (1956), we introduce the moments

$$c^{(p)}(y,t) = \int_{-\infty}^{\infty} x^p\, c(x,y,t)\, dx \quad (p = 0,1,2,\ldots), \tag{2.2}$$

and we replace equations (2.1) by the x-independent equations

$$\left.\begin{aligned} &h\partial_t c^{(p)} - \partial_y(hK_2\partial_y c^{(p)}) = p(u-\bar{u})\,hc^{(p-1)} + p(p-1)\,hK_1 c^{(p-2)},\\ \text{with}\quad &hK_2\partial_y c^{(p)} = 0 \quad \text{at} \quad y = y_-, y_+,\\ \text{and}\quad &c^{(0)} = q(y), \quad c^{(1)} = c^{(2)} = \ldots = 0 \quad \text{at} \quad t = 0. \end{aligned}\right\} \tag{2.3}$$

An important sequel is that the cross-sectionally averaged moments $\bar{c}^{(p)}(t)$ evolve in accordance with the equations

$$\partial_t \bar{c}^{(p)} = p\overline{(u-\bar{u})\,x^{(p-1)}} + p(p-1)\overline{K_1 c^{(p-2)}}, \tag{2.4}$$

where the averaging incorporates the local depth as a weight factor,

$$\text{e.g. } \bar{u} = \frac{1}{A}\int_{y_-}^{y_+} h(y)\,u(y)\,dy \quad \text{with} \quad A = \int_{y_-}^{y_+} h(y)\,dy. \tag{2.5}$$

3. Centroid displacement

Aris (1956) showed that the full solutions to equations of the form (2.3) can be expressed by means of eigenfunction expansions. Unfortunately, the algebra rapidly becomes unwieldy. Indeed, Aris only calculated the deficit variance for the special case of a uniform discharge.

Another method of solution is to take the Laplace transform of the moment equations (Chatwin 1970, 1976). Results for the centroid displacement, deficit variance, and higher moments can then be inferred from the asymptotic expansion of the Laplace transform near the origin. The simple form of the solutions obtained by Chatwin for particular discharge conditions is strongly suggestive that there should be a more direct and general derivation. Such a derivation is presented below.

Here we are concerned with the moments $\bar{c}^{(0)}$, $\bar{c}^{(1)}$, $\bar{c}^{(2)}$ at large times. Thus, from equations (2.4) we infer that we only need calculate certain integral properties for $c^{(0)}, c^{(1)}$. For example if $c_\infty^{(0)}, c_\infty^{(1)}$ denote the asymptotic values of the moments, then from equations (2.4) we have

$$\bar{c}^{(2)} = 2t\{\overline{(u-\bar{u})\,c_\infty^{(1)}} + \overline{K_1 c_\infty^{(0)}}\} + 2\overline{(u-\bar{u})\int_0^t [c^{(1)} - c_\infty^{(1)}]\,dt'} + 2\overline{K_1\int_0^t [c^{(0)} - c_\infty^{(0)}]\,dt'}. \tag{3.1}$$

It is these last two terms that contribute to the deficit variance (i.e. the influence of the early stages of the dispersion process).

From equations (2.3) we infer that a large times $c^{(0)}(y,t)$ tends toward the constant value $\bar{q}$. Also, the time-integral of $c^{(0)}-\bar{q}$ satisfies the equation

$$\left.\begin{aligned} h\partial_t\left(\int_0^t [c^{(0)}-\bar{q}]\,dt'\right)-\partial_y\left(hK_2\,\partial_y\int_0^t [c^{(0)}-\bar{q}]\,dt'\right) &= h(q-\bar{q}),\\ \text{with}\qquad hK_2\,\partial_y\int_0^t [c^{(0)}-\bar{q}]\,dt' = 0 \quad \text{at} \quad y &= y_-, y_+, \end{aligned}\right\} \tag{3.2}$$

and when $t=0$ the integral is obviously zero.

For large t the ∂_t term vanishes and so we conclude that

$$\int_0^t [c^{(0)}-\bar{q}]\,dt' \sim \bar{q}Q(y) \quad \text{for large } t, \tag{3.3}$$

where the auxiliary function $Q(y)$ satisfies the equation

$$\left.\begin{aligned} \partial_y(hK_2\,\partial_y Q) &= h(\bar{q}-q)/\bar{q},\\ hK_2\,\partial_y Q = 0 \quad \text{at} \quad y = y_-, y_+ \quad &\text{and} \quad \bar{Q} = 0. \end{aligned}\right\} \tag{3.4}$$

Physically $Q(y)$ can be interpreted as being the concentration distribution for a steady, x-independent, zero-volume source with the same discharge non-uniformity as the actual discharge. If we substitute the result (3.3) into equation (2.4) then we find that at large t the centroid displacement X is given by the formula

$$X = \bar{c}_\infty^{(1)}/\bar{c}_\infty^{(0)} = \overline{(u-\bar{u})\,Q}, \tag{3.5}$$

i.e. a weighted average of the velocity profile $u(y)$ with respect to Q.

4. Deficit variance

Proceeding to the equations for $c^{(1)}(y,t)$, we are led to pose the representation

$$c_\infty^{(1)}(y) = \bar{c}_\infty^{(1)} + \bar{q}\,g(y), \tag{4.1}$$

where the auxiliary function $g(y)$ satisfies the equation

$$\left.\begin{aligned} \partial_y(hK_2\,\partial_y g) &= h(\bar{u}-u)\\ \text{with}\qquad hK_2\,\partial_y g = 0 \quad \text{at} \quad y = y_-, y_+ \quad &\text{and} \quad \bar{g} = 0. \end{aligned}\right\} \tag{4.2}$$

This is of the same form as equation (3.4) for Q with $(\bar{q}-q)/\bar{q}$ replaced by $(\bar{u}-u)$ Physically $g(y)$ describes the dependence of the centre of gravity of the contaminant in any filament on its position across the flow (Aris 1956). Alternatively, $g(y)$ is the shape factor for longest persisting concentration variants across the flow (Taylor 1953).

Next, we infer from equation (2.3) that the time integral of $c^{(1)}-c_\infty^{(1)}$ satisfies the equations

$$\left.\begin{aligned} h\,\partial_t\left(\int_0^t [c^{(1)}-c_\infty^{(1)}]\,dt'\right)-\partial_y\left(hK_2\,\partial_y\int_0^t [c^{(1)}-c_\infty^{(1)}]\,dt'\right)&\\ = -hc_\infty^{(1)} + h(u-\bar{u})\int_0^t [c^{(0)}-\bar{q}]\,dt',&\\ \text{with}\qquad hK_2\,\partial y\int_0^t [c^{(1)}-c_\infty^{(1)}]\,dt' = 0 \quad \text{at} \quad y = y_-, y_+&\\ \text{and}\qquad \int_0^t [c^{(1)}-c_\infty^{(1)}]\,dt' = 0 \quad \text{at} \quad t = 0.& \end{aligned}\right\} \tag{4.3}$$

Making use of the above results (3.3), (3.5), we deduce that for large t

$$\int_0^t [c^{(1)} - c^{(1)}_\infty]\,dt' \sim \bar{q}R(y) \tag{4.4}$$

where the function $R(y)$ satisfies the equation

$$\left.\begin{aligned} \partial_y(hK_2\,\partial_y R) &= hg + h\{\overline{(u-\bar{u})\,Q} - (u-\bar{u})\,Q\}, \\ \text{with}\qquad hK_2\,\partial_y R &= 0 \quad \text{at} \quad y = y_-, y_+. \end{aligned}\right\} \tag{4.5}$$

If we multiply equation (4.5) by $g(y)$ and integrate by parts with respect to y from y_- to y_+, then we can derive the identity

$$\overline{(u-\bar{u})\,R} = -\overline{g^2} + \overline{(u-\bar{u})\,Qg}. \tag{4.6}$$

This enables us to eliminate the occurrence of $R(y)$ in the asymptotic version of equation (3.1)

$$\bar{c}^{(2)}/\bar{c}^{(0)} \sim 2t\{\overline{(u-\bar{u})\,g} + \bar{K}_1\} - 2\overline{g^2} + 2\overline{(u-\bar{u})\,Qg} + 2\overline{K_1 Q}. \tag{4.7}$$

The $2t$ coefficient gives us a formula for the lateral shear contribution

$$D = \overline{(u-\bar{u})\,g} \tag{4.8}$$

to the total longitudinal dispersion coefficient $D + \bar{K}_1$ (Aris 1956, equation (40)).

The variance is defined in terms of the second moment relative to the centroid of the concentration distribution:

$$V = (\bar{c}^{(2)}/\bar{c}^{(0)}) - (\bar{c}^{(1)}/\bar{c}^{(0)})^2. \tag{4.9}$$

Thus, from equations (4.7), (3.5) we conclude that the deficit variance, relative to the constant dispersion coefficient prediction $2[D + \bar{K}_1]\,t$, is given by

$$\Delta V = 2\overline{g^2} - 2\overline{(u-\bar{u})\,Qg} - 2\overline{K_1 Q} + [\overline{(u-\bar{u})\,Q}]^2. \tag{4.10}$$

For the important special case in which the discharge is uniform, we have $Q = 0$, and equation (4.10) gives the deficit variance as being $2\overline{g^2}$ in agreement with equation (3.8) of Chatwin (1970).

The centroid displacement means that a fixed distance downstream of the discharge position the contaminant cloud arrives at a time $X/\bar{u}$ ahead of the constant diffusivity prediction. In this case the appropriate measure of the deficit variance is the 'adjusted deficit variance'

$$\text{'}\Delta V\text{'} = \Delta V + 2(D + \bar{K}_1)\,X/\bar{u} \tag{4.11}$$

(i.e. the observed variance at a fixed position must account for the observed time of arrival).

5. Auxiliary equations

The usefulness of the above formula (3.4), (4.8), (4.11) for the centroid displacement X, the dispersion coefficient D, and the adjusted deficit variance 'ΔV', depends upon the ease with which we can calculate the auxiliary functions $Q(y)$, $g(y)$. In this section we present general solutions valid for arbitrary depth profiles and source distributions.

First, from equation (3.4) we note that the field equation and boundary conditions are satisfied if

$$hK_2 \partial_y Q = a_-(y) \int_y^{y_+} \frac{h(q-\bar{q})}{\bar{q}} dy' - a_+(y) \int_{y_-}^{y} \frac{h(q-\bar{q})}{\bar{q}} dy', \tag{5.1}$$

where a_-, a_+ are the fractional cross-sectional areas of the channel from the y_- and y_+ sides:

$$a_-(y) = \int_{y_-}^{y} \frac{h(y')}{A} dy', \quad a_+(y) = \int_{y}^{y_+} \frac{h(y')}{A} dy', \quad a_- + a_+ = 1. \tag{5.2}$$

Preserving the symmetry with respect to y_-, y_+ we find that a further integration of equation (5.1) yields the solution

$$Q(y) = \int_{y_-}^{y} \frac{a_-}{hK_2} \left(\int_{y'}^{y_+} \frac{h(q-\bar{q})}{\bar{q}} dy'' \right) dy' + \int_{y}^{y_+} \frac{a_+}{hK_2} \left(\int_{y_-}^{y'} \frac{h(q-\bar{q})}{\bar{q}} dy'' \right) dy'. \tag{5.3}$$

Using integration by parts and the definitions (5.2) we can verify that $\bar{Q} = 0$. To obtain the solution for $g(y)$ we simply replace $(q-\bar{q})/\bar{q}$ by $u-\bar{u}$:

$$g(y) = \int_{y_-}^{y} \frac{a_-}{hK_2} \left(\int_{y'}^{y_+} h(u-\bar{u})\, dy'' \right) dy' + \int_{y}^{y_+} \frac{a_+}{hK_2} \left(\int_{y_-}^{y'} h(u-\bar{u})\, dy'' \right) dy'. \tag{5.4}$$

Substituting the expression (5.3) for Q into equation (3.5) and performing two integrations by parts, we find that the centroid displacement is given by the formula

$$X = \frac{1}{A} \int_{y_-}^{y_+} \frac{hq}{\bar{q}} g(y)\, dy. \tag{5.5}$$

Thus, as was remarked by Aris (1956, §4) for the special case of pipe flow, when there is a point source the dependence of X upon the source position is identical to $g(y)$.

Next, substituting for $g(y)$ into equation (4.8) and integrating by parts, we obtain the equation

$$D = \frac{1}{A} \int_{y_-}^{y_+} \frac{1}{hK_2} \left[\int_{y_-}^{y} h(u-\bar{u})\, dy' \right]^2 dy. \tag{5.6}$$

This result agrees with the work of Fischer (1967) and clearly demonstrates the positivity of the shear dispersion coefficient D.

Finally, we evaluate the turbulence and shear contributions

$$-2\overline{K_1 Q} \quad \text{and} \quad -2\overline{(u-\bar{u})Qg}$$

to the deficit variance. By analogy with the definitions (5.2), we introduce the fractional turbulence and shear dispersion coefficients k_-, k_+, d_-, d_+:

$$k_-(y) = \frac{1}{A\bar{K}_1} \int_{y_-}^{y} hK_1(y')\, dy', \quad k_+(y) = \frac{1}{A\bar{K}_1} \int_{y}^{y_+} hK_1(y')\, dy', \tag{5.7}$$

$$\left.\begin{aligned} d_-(y) &= \frac{1}{AD} \int_{y_-}^{y} \frac{1}{hK_2} \left[\int_{y_-}^{y'} h(u-\bar{u})\, dy'' \right]^2 dy', \\ d_+(y) &= \frac{1}{AD} \int_{y}^{y_+} \frac{1}{hK_2} \left[\int_{y'}^{y_+} h(u-\bar{u})\, dy'' \right]^2 dy'. \end{aligned}\right\} \tag{5.8}$$

This notation enables us to write

$$\overline{K_1 Q} = \bar{K}_1 \int_{y_-}^{y_+} h \frac{(q-\bar{q})}{\bar{q}} \left\{ \int_{y_-}^{y} \frac{k_+ a_-}{hK_2} dy' + \int_{y}^{y_+} \frac{k_- a_+}{hK_2} \right\} dy, \tag{5.9}$$

and to reduce the lengthy expression for $\overline{(u-\bar{u})\,Qg}$ to the neat result

$$\overline{(u-\bar{u})\,Qg} = \frac{1}{A} \int_{y_-}^{y_+} \tfrac{1}{2} g^2(y)\, h \frac{(q-\bar{q})}{\bar{q}} dy + D \int_{y_-}^{y_+} \frac{h\,(q-\bar{q})}{\bar{q}} \left\{ \int_{y_-}^{y} \frac{d_+ a_-}{hK_2} dy' + \int_{y}^{y_+} \frac{d_- a_+}{hK_2} dy' \right\} dy. \tag{5.10}$$

6. Point discharges

For a point discharge we have the delta-function description

$$hq/\bar{q} = A\delta(y-y_0), \tag{6.1}$$

where y_0 is the discharge position. Thus, the integrals (5.5), (5.9), (5.10) can be simplified still further and we find that the adjusted deficit variance (4.11) is given by

$$\begin{aligned} \text{`}\Delta V\text{'} = {} & 3\overline{g^2} + 2A(D+\bar{K}_1) \int_{y_-}^{y_+} \frac{a_- a_+}{hK_2} dy \\ & - 2AD \left\{ \int_{y_-}^{y_0} \frac{d_+ a_-}{hK_2} dy' + \int_{y_0}^{y_+} \frac{d_- a_+}{hK_2} dy' \right\} \\ & - 2A\bar{K}_1 \left\{ \int_{y_-}^{y_0} \frac{k_+ a_-}{hK_2} dy' + \int_{y_0}^{y_+} \frac{k_- a_+}{hK_2} dy' \right\} \\ & + 2(D+\bar{K}_1) A \left\{ \int_{y_-}^{y_0} \frac{a_-}{hK_2} \left(\int_{y'}^{y_+} \frac{h(u-\bar{u})}{A\bar{u}} dy'' \right) dy' \right. \\ & \left. + \int_{y_0}^{y_+} \frac{a_+}{hK_2} \left(\int_{y_-}^{y'} \frac{h(u-\bar{u})}{A\bar{u}} dy'' \right) dy' \right\}. \end{aligned} \tag{6.2}$$

It might be thought that when averaged with respect to different discharge positions the above formula should yield the uniform discharge result $2\overline{g^2}$. However, as a consequence of the centroid displacement $g(y_0)$, the average of 'ΔV' is actually $3\overline{g^2}$.

As we vary the discharge position we have

$$\begin{aligned} \frac{d}{dy_0} \text{`}\Delta V\text{'} &= \frac{2A}{hK_2} \left\{ Dd_- + \bar{K}_1 k_- - (D+\bar{K}_1) a_- - (D+\bar{K}_1) \int_{y_-}^{y} \frac{h(u-\bar{u})}{A\bar{u}} dy' \right\} \\ &= -\frac{2A}{hK_2} \left\{ Dd_+ + \bar{K}_1 k_+ - (D+\bar{K}_1) a_+ - (D+\bar{K}_1) \int_{y}^{y_+} \frac{h(u-\bar{u})}{A\bar{u}} dy' \right\}. \end{aligned} \tag{6.3}$$

At the respective sides of the channel the $-$ and $+$ functions are zero. Thus, the adjusted deficit variance has an extremum when the discharge is at either bank.

To investigate the nature of the extremum we take a further derivative

$$\frac{1}{2} \frac{d}{dy_0} \left(hK_2 \frac{d}{dy_0} \text{`}\Delta V\text{'} \right) = \frac{1}{hK_2} \left[\int_{y_-}^{y} h(u-\bar{u})\, dy' \right]^2 + h\{K_1(y) - (D+\bar{K}_1)\, u/\bar{u}\}. \tag{6.4}$$

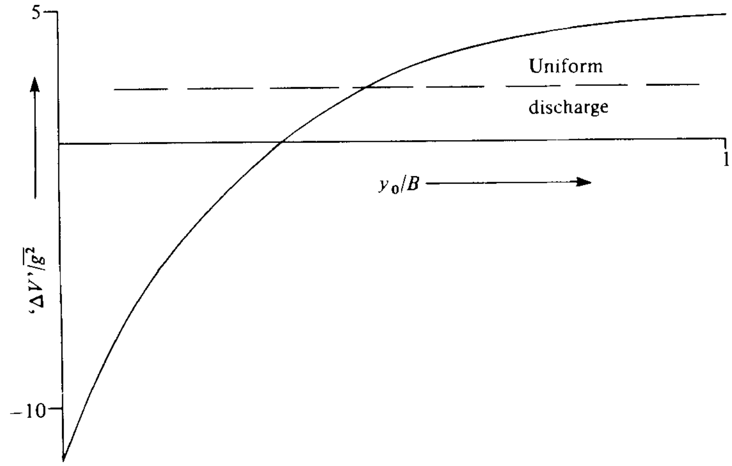

FIGURE 2. The adjusted deficit variance as a function of discharge position for a triangular channel.

If the water depth is non-zero at the sides, then for the second derivative of 'ΔV' to be negative it suffices that $u/\bar{u}$ should exceed $K_1/(D+\bar{K}_1)$. This is a minor constraint since the shear dispersion coefficient D associated with the lateral shear is usually several orders of magnitude greater than the vertical shear coefficient K_1 (Fisher 1967). Hence, in this case the adjusted deficit variance is a maximum, and the channel sides are undesirable locations for a sudden contaminant discharge.

Usually the water depth tends to zero at the sides and a more careful argument is needed. The local turbulent diffusivities are proportional to the product of the local water depth and longitudinal velocity (Elder 1959). This implies that the diffusivities vary as the three-halves power of the water depth, and the longitudinal velocity as the square-root of the depth:

$$K_2 = h^{\frac{3}{2}}\bar{h}\bar{K}_2/\overline{h^{\frac{5}{2}}}, \quad u = h^{\frac{1}{2}}\bar{h}\bar{u}/\overline{h^{\frac{3}{2}}}. \tag{6.5}$$

Using these representations in equation (6.4) and taking the limit as the water depth tends to zero, we deduce that the adjusted deficit variance is at a minimum provided that the depth gradient h' satisfies the inequality

$$(h')^2 < \frac{\overline{h^{\frac{5}{2}}}\,\overline{h^{\frac{3}{2}}}\,\bar{u}^2}{4\bar{h}^2\bar{K}_2(D+\bar{K}_1)}. \tag{6.6}$$

Thus, for sufficiently gentle sloping banks we have the important conclusion that the channel sides are the best possible sites for a sudden discharge. For steeper-sided channels the conclusion is reversed, as indeed it must be in order to be compatible with the earlier results for vertically-sided channels.

7. Triangular channel

In order to illustrate the dependence of the adjusted deficit variance (6.2) upon the source position, we take the depth profile to be an isosceles triangle with width $2B$ and maximum depth H:

$$h(y) = (y/B)\,H \quad \text{for} \quad 0 < y < B, \tag{7.1}$$

with a symmetrical expression valid for $B < y < 2B$. For the diffusivity model we take

$$K_1 = 6hu_*, \quad K_2 = 0{\cdot}15hu_*, \tag{7.2}$$

where $u_*(y)$ is the friction velocity (Fischer 1973). If we assume that there is a constant ratio γ (about 0.1) between $u_*(y)$ and $u(y)$, then it follows that the diffusivities and velocity profile take the form given by equations (6.5):

$$K_2 = 0{\cdot}15\gamma H\bar{u}\tfrac{5}{4}(y/B)^{\frac{3}{2}}, \quad u = \bar{u}\tfrac{5}{4}(y/B)^{\frac{1}{2}}. \tag{7.3}$$

In particular, it follows that $\qquad \bar{K}_1 = 6\gamma H\bar{u}(\tfrac{5}{4}). \qquad (7.4)$

It is now a straightforward but lengthy task to evaluate all the necessary integrals (for other topographies the results would be numerically similar but not as simple):

$$a_- = \tfrac{1}{2}(y/B)^2, \tag{7.5}$$

$$\int_{y_-}^{y} h(u-\bar{u})\,dy = \tfrac{1}{2}BH\bar{u}[(y/B)^{\frac{5}{2}} - (y/B)^2], \tag{7.6}$$

$$g = \frac{B^2}{0{\cdot}15\gamma H}[-\tfrac{28}{75} + \tfrac{4}{5}(y/B)^{\frac{1}{2}} - \tfrac{2}{5}(y/B)], \tag{7.7}$$

$$D = \frac{4}{525}\,\frac{B^2\bar{u}}{0{\cdot}15\gamma H} = \frac{32}{4725}\left(\frac{B}{\gamma H}\right)^2 \bar{K}_1, \tag{7.8}$$

$$2d_- = 15(y/B)^{\frac{7}{2}} - 35(y/B)^3 + 21(y/B)^{\frac{5}{2}}, \tag{7.9}$$

$$\overline{g^2} = \frac{31}{14}\left(\frac{B^2 2}{0{\cdot}15\gamma H 75}\right)^2, \tag{7.10}$$

$$\text{‘}\Delta V\text{’} = \overline{g^2}\cdot\tfrac{1}{31}\{-357 + 100[9(y_0/B)^2 - 28(y_0/B)^{\frac{3}{2}} + 24(y_0/B)]\}. \tag{7.11}$$

We recall that y_0 denotes the position of the point discharge. Equation (7.8) explicitly demonstrates that D is much greater than K_1 (Fischer 1967). For example,

$$\text{if} \quad B/H = 9 \quad \text{and} \quad \gamma = 0{\cdot}1 \quad \text{then} \quad D/\bar{K}_1 = 55. \tag{7.12}$$

Thus, the key result (7.11) for the adjusted deficit variance has been simplified by the neglect of all $\bar{K}_1$ terms.

A remarkable outcome of the above analysis is that when the discharge is close to the bank the adjusted deficit variance is negative. This means that at large distances downstream of the discharge, the variance actually exceeds the constant dispersion coefficient prediction. The magnitude of the extra variance is also remarkable. The e-folding distance for the decay of transverse concentration variations is

$$X_e = 2{\cdot}1B^2/\gamma H = 7{\cdot}9(\overline{g^2})^{\frac{1}{2}} \tag{7.13}$$

(Smith 1979, equation (2.16)). Cross-sectional mixing can be regarded as having been established after about three times this distance. Depending upon whether the discharge is at the bank, cross-sectionally uniform, or at the centre-line, we find that the total variance $6DX_e/\bar{u} - \text{‘}\Delta V\text{’}$ at this stage is given by

$$20{\cdot}6\overline{g^2}, \quad 7{\cdot}1\overline{g^2}, \quad 4{\cdot}5\overline{g^2}. \tag{7.14}$$

When converted into concentration predictions, by taking the inverse square root of the variance (see appendix), there is over a factor of two between the extremes. Thus, there is a considerable premium in choosing the right position for a contaminant discharge.

The inequality (6.6) shows that if the channel bank is sufficiently steep, then it ceases to be the best site for a point discharge. An estimate of the necessary steepness can be obtained by using the above results (7.4), (7.8) for $\bar{K}_2$ and D:

$$(h') < 21(H/B)^2. \tag{7.15}$$

Thus, it is only for extremely steep-sided channels that the inequality is violated. In this unusual circumstance the best position for a point discharge could be determined by finding the roots of equation (6.3). However, as a general rule we conclude that in narrow rivers or estuaries the best that can be done with a contaminant release is to follow the age-old practice of making the discharge at the bank.

The author wishes to thank B.P. and the Royal Society for financial support.

Appendix. Concentration distribution

In practice it is not the cross-sectionally averaged concentration $\bar{c}$ which is monitored, but rather the concentration at some position close to the side of the channel. Thus, the observed time of arrival of the contaminant cloud and the observed variance will differ from the predictions made in the above paper. This leads to the crucial question as to whether the concept of the best discharge site depends upon the mode of observation? Fortunately, it is easy to demonstrate that the relative performance of different discharge sites remains the same independently of the monitoring position across the flow.

For a sudden discharge the concentration distribution eventually becomes Gaussian. The details of the approach to this asymptotic state have been investigated by Chatwin (1970). In terms of the variance V and the centroid displacement X, Chatwin's equation (2.19) can be written

$$\left.\begin{aligned} c &= \frac{\bar{q}}{A(2\pi V)^{\frac{1}{2}}}\exp(-\tfrac{1}{2}z^2)\left\{1+V^{-\frac{1}{2}}\left[g(y)\,He_1+\frac{1}{2}\frac{\overline{g^2(u-\bar{u})}}{\overline{g(u-\bar{u})}}He_3\right]+O(V^{-\frac{3}{2}})\right\} \\ \text{with}\quad & He_1(z)=z,\quad He_3(z)=z^3-3z, \\ & z=(x-\bar{u}t-X)/V^{\frac{1}{2}}, \\ & V=2[D+\bar{K}_1]t-\Delta V. \end{aligned}\right\} \tag{A 1}$$

Here $g(y)$ is the shape factor for the concentration variations across the flow (see equations (4.2), (5.4)). We note that the coefficient of the skewness term He_3 involves the velocity profile $(u-\bar{u})$ and the shape factor $g(y)$, and does not depend upon the initial conditions. Thus, at large times the influence of the discharge site is via the deficit variance ΔV and the centroid displacement X.

From equation (A 1) it follows that at positions y_m across the flow the apparent centroid position and variance are given by

$$\bar{u}t+X+g(y_m), \quad V-g(y_m)^2. \tag{A 2}$$

This means that the maximum concentration is experienced at a time $g(y_m)/\bar{u}$ different from that for $\bar{c}$. The variance at this time of maximum concentration is given by

$$2[D+\bar{K}_1]\,x_m/\bar{u} - \Delta V - 2[D+\bar{K}_1]\,X/\bar{u} \qquad \text{(A 3)}$$
$$- g(y_m)\,[2(D+\bar{K}_1)/\bar{u} + g(y_m)],$$

where x_m is the monitoring position along the channel. Clearly, the dependence upon the discharge conditions (i.e. upon ΔV and X) is independent of the observation position. Varying y_m is equivalent to measuring the cross-sectionally averaged concentration at positions x_m displaced upstream or downstream accordingly as $g(y_m)$ is positive or negative.

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